Vector Calculus

14 lectures given in the autumn term to second year physicists as part of the Maths II course

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About the course

As the name implies, vector calculus is a combination of vector algebra and calculus. It is an elegant branch of mathematics that is extremely useful to physicists as it is the mathematics of fields. You will already have encountered electric, magnetic and gravitational fields, but you are soon to come across many more. Vector calculus gives you the mathematical tools to manipulate fields, to do calculations involving fields, to describe their properties and to characterise them precisely. It is an important branch of mathematical physics.

As we live in a three-dimensional world, the variables we will deal with are x, y, z, i.e. three-dimensional space. Towards the end of the course, *time* will be introduced as an extra variable – not everything stands still and we need to be able to deal with things that move or change with time.

What you need to know

It is assumed that you are already familiar with both vector algebra and the calculus of functions of more than one variable. In particular, make sure you know about:

Vector algebra: Cartesian components, dot and cross products, triple products.
Calculus: partial differentiation, total differential, differential operators, multiple integrals.
Coordinate systems: Cartesian, spherical polar, cylindrical polar.

Mathematics provides the framework to make difficult things easy.

http://www-users.york.ac.uk/~pm1/PMweb/teaching.htm

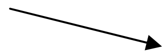
Revision

Vector algebra

A vector is a quantity that has both magnitude and direction. It is useful in physics because it can be used to represent velocity, acceleration, momentum, force, position, displacement and many other quantities. The following is a review of the essentials of vector algebra that will be used in this term's mathematics course.

Representation

The algebraic symbol for a vector may be written in bold as **a**, or underlined as $\underline{a}$. In a diagram it is represented by an arrow:



The magnitude of **a** can be written as *a*, which is a scalar quantity and its direction can be given by a unit vector **n**. The magnitude of **n** is unity, hence its name. It is a dimensionless quantity and is used only to define a direction.

We can therefore write **a** as a**n** where the magnitude and direction are given separately.

Cartesian components

A vector is often described in terms of its Cartesian components, i.e. the components of the vector parallel to the *x*-, *y*- and *z*-axes.

The directions of the axes are given by the unit vectors **i**, **j**, and **k** so that a vector may be written in terms of its components as $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

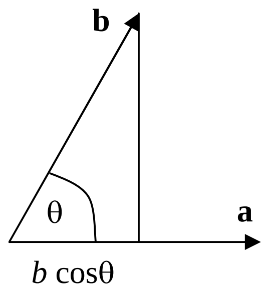
Pythagoras's theorem gives the magnitude of the vector as $a = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Dot product

There are two ways of multiplying vectors, depending upon the context. The dot product of **a** and **b** is written as **a.b** and the result of the multiplication is a scalar. It is therefore also called a **scalar product**.

The result of the multiplication is $\mathbf{a.b} = a\,b\cos\theta$ where θ is the angle between the vectors. This definition shows that $\mathbf{a.b} = \mathbf{b.a}$, i.e. the vectors commute.

It can be seen in the diagram that $b\cos\theta$ is the projection of **b** on **a**. This is often a useful way of thinking about scalar products, i.e. as *a* multiplied by the projection of **b**. It is also used to resolve one vector in the direction of another.



If the vectors are orthogonal to one another, i.e. at right angles, then $\mathbf{a.b} = 0$

Also, the definition of the dot product leads to $\mathbf{a.a} = a^2$

The dot products of the unit vectors **i**, **j** and **k** are therefore:

$$\mathbf{i.i} = \mathbf{j.j} = \mathbf{k.k} = 1;\quad \mathbf{i.j} = \mathbf{j.i} = 0;\quad \mathbf{i.k} = \mathbf{k.i} = 0;\quad \mathbf{j.k} = \mathbf{k.j} = 0$$

When the vectors are expressed in terms of their Cartesian components, so that $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, the dot product becomes $(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$. Multiplying out the brackets and using the dot products of the unit vectors given above results in

$$\mathbf{a.b} = a_1\,b_1 + a_2\,b_2 + a_3\,b_3$$

Cross product

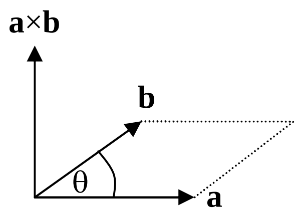
The other way of multiplying vectors is the cross product, written as $\mathbf{a}\times\mathbf{b}$ which results in a vector. It is therefore also called a **vector product**.

The vector product is defined by $\mathbf{a}\times\mathbf{b} = ab\sin\theta\,\mathbf{n}$

where θ is the angle between the vectors and **n** is a unit vector perpendicular to both **a** and **b** such that **a**, **b**, **n** form a right-handed set.

This definition requires that $\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a}$ since reversing the order of the vectors changes the sense of **n**, i.e. it points in exactly the opposite direction. You must always be careful of the order of the vectors when dealing with cross products.

A geometric interpretation of the cross product is shown in the diagram. The two vectors **a** and **b** define the parallelogram and the magnitude of $\mathbf{a}\times\mathbf{b}$ is its area. The direction of $\mathbf{a}\times\mathbf{b}$ is normal to the plane of the parallelogram.



If the two vectors are parallel, the cross product gives the null vector. In particular, $\mathbf{a}\times\mathbf{a} = \mathbf{0}$

The cross products of the unit vectors **i**, **j** and **k** are:

$$\mathbf{i}\times\mathbf{j} = \mathbf{k};\quad \mathbf{j}\times\mathbf{k} = \mathbf{i};\quad \mathbf{k}\times\mathbf{i} = \mathbf{j};\qquad \mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$$

Note that, in the first three relationships, the vectors are always in the same cyclic order.

With $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and using the above relationships, the cross product becomes

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = (a_2b_3-a_3b_2)\mathbf{i} + (a_3b_1-a_1b_3)\mathbf{j} + (a_1b_2-a_2b_1)\mathbf{k}$$

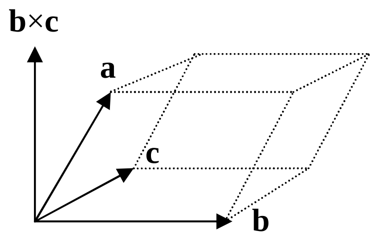
Fortunately, there is an easier way of expressing this: $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Expansion of the determinant gives the same result as above.

Scalar triple product

Triple products of vectors frequently arise. The scalar triple product, written as $\mathbf{a}.\mathbf{b}\times\mathbf{c}$, results in a scalar quantity, hence its name. To make mathematical sense, the cross product must be evaluated first, giving a vector which is dotted with **a**.

A geometrical interpretation of the scalar triple product is shown in the diagram. The three vectors define a parallelepiped. The magnitude of $\mathbf{b}\times\mathbf{c}$ gives the area of the base and its direction is normal to the base. The dot product with **a** therefore resolves **a** in the direction of $\mathbf{b}\times\mathbf{c}$, giving the vertical height. Area of base multiplied by the height gives the volume of the parallelepiped.



We therefore have the result that $\mathbf{a}.\mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{b}.\mathbf{c}$ since the same three vectors are involved and therefore give the same volume.

Similarly: $\mathbf{a}.\mathbf{b}\times\mathbf{c} = \mathbf{b}.\mathbf{c}\times\mathbf{a} = \mathbf{c}.\mathbf{a}\times\mathbf{b} = -\mathbf{c}.\mathbf{b}\times\mathbf{a} = -\mathbf{b}.\mathbf{a}\times\mathbf{c} = -\mathbf{a}.\mathbf{c}\times\mathbf{b}$

Clearly, if two of the vectors are parallel, the scalar triple product must have a value of zero.

Vector triple product

The other triple product that arises is the vector triple product, so called because it results in a vector quantity.

It is written as $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ or $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$. Note that the brackets are necessary to indicate which product is performed first because $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} \neq \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

While providing a very compact expression, vector triple products are awkward to deal with. For the purposes of algebraic manipulation, they are nearly always changed to an alternative expression using the standard identity:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a.c})\,\mathbf{b} - (\mathbf{a.b})\,\mathbf{c}$$

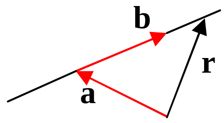
Straight line

Straight lines in 3D space are most conveniently represented using vectors.

The position of the line is given by specifying a point on it, e.g. the point **a**, and its direction is given as parallel to the vector **b**:

$$\mathbf{r}(\lambda) = \mathbf{a} + \lambda\,\mathbf{b}$$

where **r** is the position vector of a point on the line and the scalar variable λ moves the point along the line.



If the line is defined as going through the two points **a** and **b**, then the vector **b-a** is in the direction of the line and its equation is therefore: $\mathbf{r}(\lambda) = \mathbf{a} + \lambda\,(\mathbf{b}-\mathbf{a})$

Space curve

A space curve is a curve in 3D space which may be used, for example, to describe the path of a particle in a force field.

An example of a space curve is: $\mathbf{r}(\theta) = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j} + b\,\theta\,\mathbf{k}$

where $\mathbf{r}(\theta)$ is the position vector of a point on the curve as a function of θ. As θ varies, the position vector traces out the curve.

In this example, the **i** and **j** components trace out a circle of radius a as θ varies. In addition to this, the **k** component varies linearly to move the particle along the z-axis, making the space curve into a helix lying along the z direction with a pitch of 2πb.

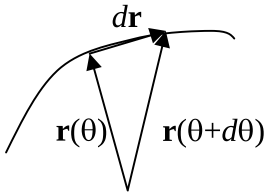
Differentiation of vectors

If the vector is defined as a function of one or more variables, the possibility arises of differentiating the vector function. This is a common operation in vector calculus.

Taking the above space curve as an example, the derivative with respect to θ is obtained by differentiating each component separately:

$$\frac{d\mathbf{r}}{d\theta} = -a\sin\theta\,\mathbf{i} + a\cos\theta\,\mathbf{j} + b\,\mathbf{k}$$

which can be rewritten as: $d\mathbf{r} = (-a\sin\theta\,\mathbf{i} + a\cos\theta\,\mathbf{j} + b\,\mathbf{k})\,d\theta$



The infinitesimal vector $d\mathbf{r}$ is the displacement required to move from $\mathbf{r}(\theta)$ to $\mathbf{r}(\theta+d\theta)$, i.e. $\mathbf{r}(\theta+d\theta) = \mathbf{r}(\theta) + d\mathbf{r}$

Since both $\mathbf{r}(\theta)$ and $\mathbf{r}(\theta+d\theta)$ lie on the space curve, the vector $d\mathbf{r}$ must lie along the curve, so it is in the direction of the tangent. This makes it possible to move along a space curve in a series of infinitesimal steps $d\mathbf{r}$.

Vector calculus

Scalar field

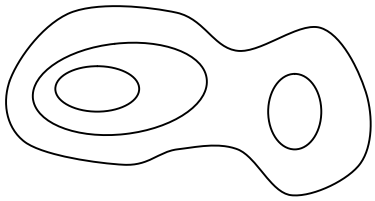
Many scalar quantities have only a single value, e.g. mass of an electron, specific heat of copper, speed of light. However, the value of a physical quantity may depend upon position such as the air temperature in a large hall or the height above mean sea level of an area on the Earth's surface.

The association of a particular value of a physical quantity with each point in a region of space is said to constitute a **field**. When the physical quantity is a scalar, the field is called a **scalar field**.

Definition If to each point in a region of space there corresponds a scalar quantity ϕ, then $\phi(x,y,z)$ is known as a **scalar field**.

Examples Temperature at each point within the Earth's surface.
Electric potential at every point in an electron optical system.
$\phi(x,y,z) = x^3y - z^2$ defines a scalar field.

Representation Scalar fields are best represented as contour maps in 2 or 3 dimensions.



Vector field

Just as there are scalar fields, there are also vector fields. The velocity of a boat on a river can be represented by a single vector, but the velocity of the water in the river can not. The water velocity depends upon where it is measured.

Definition If to each point in a region of space there corresponds a vector quantity $\mathbf{V}$, then $\mathbf{V}(x,y,z)$ is known as a **vector field**.

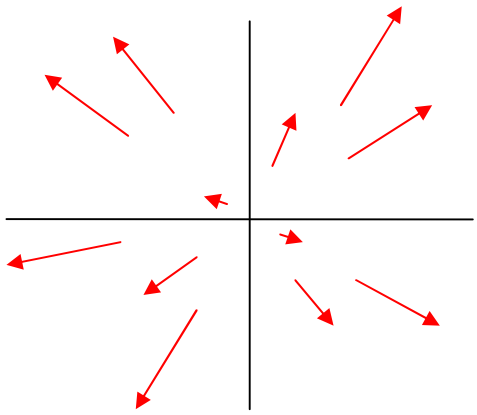
Examples Velocity at every point in a moving fluid.
Magnetic field at every point in an electron microscope.
$\mathbf{V}(x,y,z) = xy^2\mathbf{i} - 2yz^3\mathbf{j} + x^2z\mathbf{k}$ defines a vector field.

Before deciding on a good representation for a vector field, let us graph the field $\mathbf{V}(x,y) = x\mathbf{i} + y\mathbf{j}$.

The magnitude of the vector is $\sqrt{x^2 + y^2}$, which corresponds to distance from the origin.

The direction of the vector is $\arctan\left(y/x\right)$, which always points directly away from the origin.

Placing a vector at each point in the diagram clearly gives a very clumsy picture of the field, although this is sometimes used.



Representation Since the quantity that distinguishes a vector field from a scalar field is its direction, it is this which is plotted. This gives a clearer representation than when the magnitude is plotted as well. Thus, a vector field is most conveniently represented using **field lines**.

Field line A curve whose tangent vector at each point is in the direction of the vector field at that point is known as a **field line**. (You may also have used the terms *line of force*, *stream line* or *flow line*.)

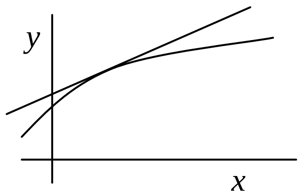
Having introduced fields, we are now going to look at three main measures of field characteristics which enable us to do many important calculations of field properties.

Gradient of a scalar field

The first of the important field characteristics to be considered is the **gradient** of a scalar field.

For a function of a single variable we have $\frac{dy}{dx} = gradient$

which can be rearranged as $dy = gradient \times dx$



This expression relates the amount by which the function changes, *dy*, for a small shift in the value of *x*, *dx*.

Now consider a scalar field $\phi(x,y,z)$ and find the change in the value of ϕ for an infinitesimal vector displacement **dl**.

Comparison with the above relationship suggests we can write

$$d\phi = gradient\ .\ \mathbf{dl}$$

where it is clear that *gradient* must be a vector quantity and that a dot product with **dl** is required to produce the scalar quantity $d\phi$.

This is the gradient of the scalar field ϕ and the standard way of writing it is

$$\boxed{d\phi = \text{grad}\phi\ .\ \mathbf{dl}}$$

To determine an expression for gradφ in terms of *x*, *y* and *z*, we can express **dl** as

$$\mathbf{dl} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$$

and the total differential of ϕ is $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

Putting these into the expression for $d\phi$ requires gradϕ to be

$$\boxed{\text{grad}\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}}$$

Note that ϕ is a **scalar field** and that gradϕ is a **vector field**. However, not all vector fields can be expressed in terms of the gradient of a scalar field in this way.

Since $d\phi$ = gradφ **. dl**, the maximum value of $d\phi$ occurs when gradϕ and **dl** are parallel. (Remember that **a.b** = $ab\cos\theta$, which has a maximum when $\theta = 0$, i.e. when the vectors are parallel.) The maximum rate of increase of ϕ is therefore in the direction of gradϕ.

We now have the important results that:

Direction of gradϕ = direction of maximum rate of increase of ϕ.

Magnitude of gradϕ = maximum rate of increase of ϕ.

Vector differential operator ∇ 'del'

It is convenient to express the gradient of a scalar field in terms of the vector differential operator:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Compare with the operator $\frac{d}{dx}$ which gives the gradient of $f(x)$.

The gradient of the scalar field $\phi(x,y,z)$ can now be written as

$$\text{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Mathematical example

If $\phi(x,y,z) = 3x^2y - y^3z^2$, find grad$\phi$ at the point (1, -2, -1).

Solution:

$$\nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left(3x^2y - y^3z^2\right)$$

$$= \mathbf{i}\frac{\partial}{\partial x}\left(3x^2y - y^3z^2\right) + \mathbf{j}\frac{\partial}{\partial y}\left(3x^2y - y^3z^2\right) + \mathbf{k}\frac{\partial}{\partial z}\left(3x^2y - y^3z^2\right)$$

$$\therefore \nabla\phi = 6xy\,\mathbf{i} + \left(3x^2 - 3y^2z^2\right)\mathbf{j} - 2y^3z\,\mathbf{k}$$

This is the gradient of $\phi(x,y,z)$, so at the point (1, -2, -1) we have

$$\nabla\phi = -12\,\mathbf{i} - 9\,\mathbf{j} - 16\,\mathbf{k}$$

Manipulations of ∇

Here are some examples of algebraic and differential operations that may be carried out using the ∇ operator:

$\nabla(\theta + \phi) = \nabla\theta + \nabla\phi$ – normal rules of calculus apply

$\nabla(\theta\phi) = (\nabla\theta)\phi + \theta\nabla\phi$ – the product rule

An example which will be useful later is:

Evaluate $\nabla\left(\frac{1}{r}\right)$ where $\mathbf{r}$ = x $\mathbf{i}$ + y $\mathbf{j}$ + z $\mathbf{k}$ i.e. $r = \left(x^2 + y^2 + z^2\right)^{1/2}$

Since r is symmetric in x, y and z, it is sufficient only to determine $\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$:

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{-1/2} = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2} 2x = -\frac{x}{r^3}$$

We can now use this result to determine the other components of $\nabla\left(\frac{1}{r}\right)$:

$$\therefore \nabla\left(\frac{1}{r}\right) = -\frac{x}{r^3}\mathbf{i} - \frac{y}{r^3}\mathbf{j} - \frac{z}{r^3}\mathbf{k} = -\frac{\mathbf{r}}{r^3}$$

Another example which illustrates the mathematics of the ∇ operator is to evaluate $\nabla(\mathbf{a.r})$ where **r** is defined above and **a** = a_1 **i** + a_2 **j** + a_3 **k** is a constant:

$$\nabla(\mathbf{a.r}) = \nabla(a_1\, x + a_2\, y + a_3\, z) = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k} = \mathbf{a}$$

Directional derivative of a scalar field

At any point in a scalar field ϕ, the rate of change of ϕ will depend upon direction. The rate of change given by $\nabla\phi$ corresponds to the direction and magnitude of the maximum gradient at that point.

The component of $\nabla\phi$ in the direction of a unit vector **n** is given by $\nabla\boldsymbol{\varphi}.\mathbf{n}$, which is the **directional derivative** of ϕ in the direction of **n**. It gives the rate of change of ϕ at (x, y, z) in the direction of **n**.

It is easily derived as follows: $d\phi = \text{grad}\phi.\mathbf{dl}$

Express **dl** as **n** *dl* then $d\phi = \text{grad}\phi.\mathbf{n}\; dl$

So that $\dfrac{d\phi}{dl} = \text{grad}\phi.\mathbf{n} = \nabla\phi.\mathbf{n}$

Physical examples

1. Electric field Work done against an electric field **E** on moving a unit charge a vector distance **dl** is **–E.dl**. By definition, this is equal to *dV*, the potential difference between the ends of the vector **dl**. (Remember that work done *against* a field is negative, work done *by* the field is positive.)

From the physical definition of potential: $dV = -\mathbf{E.dl}$

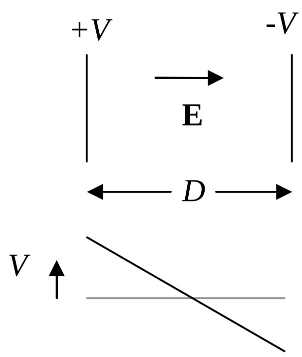
For a scalar field ϕ, we already have: $d\phi = \text{grad}\phi.\mathbf{dl}$

$\therefore$ from the mathematical definition of grad: $dV = \text{grad}V.\mathbf{dl}$

Comparison of these expressions gives: $\boxed{\mathbf{E} = -\,\text{grad}V}$

Electric potential, *V*, is a scalar field and the electric field **E**, the gradient of *V*, is the corresponding vector field. The negative sign in the above expression comes about because the direction of **E** is from a region of high potential to one of low potential, but grad*V* is in the direction of increasing *V*.

That the electric field is the negative of the gradient of the electric potential is easily seen in a one-dimensional example.



For the simple field in the diagram: $E = \dfrac{2V}{D}\,\text{volts/metre}$

The slope (gradient) of the graph is $-\dfrac{2V}{D}$

So the magnitude and direction of the field is $\mathbf{E} = +\dfrac{2V}{D}\mathbf{i}$

Note that $\text{grad}V = \dfrac{\partial V}{\partial x}\mathbf{i} + \dfrac{\partial V}{\partial y}\mathbf{j} + \dfrac{\partial V}{\partial z}\mathbf{k}$

For V in volts and x, y, z in metres, the dimensions of grad$V$ must be volts / metre.

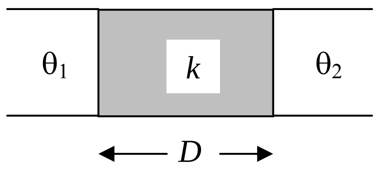
2. Heat conduction

Fourier's first law of heat conduction is: $\boxed{\mathbf{Q} = -\, k \,\text{grad}\theta}$

where $\mathbf{Q}$ = quantity of heat/unit time/ unit area flowing in the direction of $\mathbf{Q}$.
k = thermal conductivity
θ = temperature

Again, we have an elegant relationship between a scalar field θ and a vector field $\mathbf{Q}$. The negative sign arises because heat flows from high temperature to low, but gradθ is in the direction of increasing temperature.

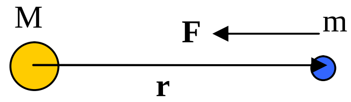
A formula in heat conduction you should already be familiar with is:



Heat / unit time $= k \times area \times \dfrac{\theta_1 - \theta_2}{D}$

This can be seen as Fourier's law in one dimension.

3. Gravity



From Newton's law of gravitation, the force exerted on mass m by mass M is: $F = \dfrac{G\,M\,m}{r^2}$

With vectors, this expression is: $\mathbf{F} = -\dfrac{G\,M\,m\,\mathbf{r}}{r^3}$

where the negative sign is necessary since $\mathbf{F}$ and $\mathbf{r}$ are clearly in opposite directions.

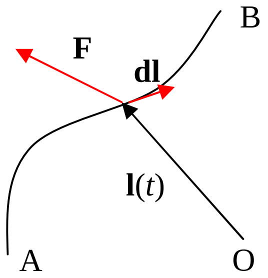
Now gravitational potential is $V = -\dfrac{G\,M\,m}{r}$

From previous results, we obtain grad$V = -G\,M\,m\nabla\left(\dfrac{1}{r}\right) = \dfrac{G\,M\,m\,\mathbf{r}}{r^3}$

Hence, we have the relationship $\boxed{\mathbf{F} = -\,\text{grad}V}$

and the gravitational force field is minus the gradient of the gravitational potential.

Line integral



Here is a very important calculation on a scalar field.

Consider a particle constrained to move along the line AB in the force field $\mathbf{F}$.

The line AB is represented by the space curve $\mathbf{l}(t)$.

The work done by the field in moving the particle an infinitesimal distance $\mathbf{dl}$ along the line is $dW = \mathbf{F.dl}$

Therefore, the total work done by the field in moving the particle from A to B is: $W = \int_A^B \mathbf{F.dl}$

This is the **line integral** of $\mathbf{F}$ along the curve AB.

If the integral is performed round a closed path, returning to the starting point, it is written as $\oint_C \mathbf{F.dl}$ where C labels the path.

The value of a line integral will normally depend upon the positions of A and B and also upon the path between them. We will see shortly how line integrals are evaluated.

Conservative field and scalar potential

A special relationship between a scalar and a vector field is that the vector field **F** is obtained as the gradient of the scalar field ϕ, i.e. $\mathbf{F} = -\,\text{grad}\phi$. In this case, **F** is known as a **conservative field** and ϕ is its **scalar potential**.

An important property of a conservative field, **F**, is that **the line integral of F between any two points is independent of the path between the points**. This is proved as follows:

$$\text{Work done against the field} = -\int_A^B \mathbf{F.dl} = \int_A^B \text{grad}\phi\mathbf{.dl} = \int_A^B d\phi = \phi_B - \phi_A$$

i.e. the work depends only upon the potential difference between the ends of the path.

It is clear from this that the line integral round a closed path in a conservative field must be zero: $\oint_C \mathbf{F.dl} = 0$ for any closed path C, if and only if $\mathbf{F} = -\,\text{grad}\phi$.

Examples of conservative fields

A **gravitational field** is conservative since the gravitational potential (or potential energy) of an object depends only upon its distance from the centre of the Earth, i.e. work done depends only upon difference in height and is independent of the path.

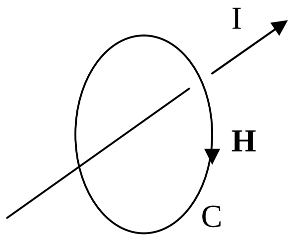
Since an **electrostatic field** is given by $\mathbf{E} = -\,\text{grad}V$, **E** must be conservative. Work done is a function only of potential difference between start and end points and not upon the path.

Non-conservative fields

Friction – the direction of the force field is always opposite to the direction of motion, i.e. $\oint \mathbf{F.dl} \neq 0$

A **magnetic field** is not conservative since Ampère's law gives:

$$\oint_C \mathbf{H.dl} = I \neq 0$$



work done in taking a unit magnetic pole round a closed loop in a magnetic field = current through the loop

A particle taken round a closed loop in the appropriate direction will gain energy and therefore can do work. This occurs in an electric motor.

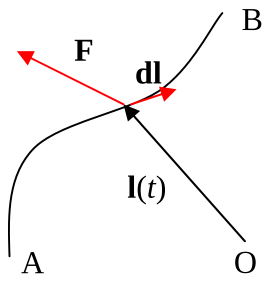
Comment

A conservative field is so-called because a particle interacting with a conservative force field, for example, will have its energy conserved. There is no energy interchange with the field and the sum of potential and kinetic energy of the particle is constant. You should already be aware of this for gravitational and electric fields. A consequence of this is that neither gravitational nor electrostatic motors can exist.

In an electric motor, the electricity is used to produce a magnetic field which is not conservative and from which energy can be extracted.

The generation of hydroelectricity does not extract energy from the Earth's gravitational field – it requires energy from the Sun to provide the water in the upper reservoir.

Evaluation of line integrals



Line integrals have already been defined. We will see now how they are evaluated.

$\int_A^B \mathbf{F.dl}$ represents the work done by the force field **F** in moving a particle along the line from A to B.

The recipe for evaluating line integrals is as follows. An actual example will be given in the next section.

1. Obtain an expression for the space curve AB, for example: $\mathbf{l}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$
2. Differentiate to obtain **dl**: $\mathbf{dl} = \left(\frac{df_1}{dt}\mathbf{i} + \frac{df_2}{dt}\mathbf{j} + \frac{df_3}{dt}\mathbf{k}\right)dt$
3. Obtain an expression for the vector field: $\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$
4. Substitute for x, y, z from 1. to obtain $\mathbf{F}(t)$, i.e. $x = f_1(t)$, $y = f_2(t)$ and $z = f_3(t)$. This makes **F** a function of t only, which is the value of the force field along the path – it doesn't matter what the force field is elsewhere: $\mathbf{F}(t) = F_1(t)\mathbf{i} + F_2(t)\mathbf{j} + F_3(t)\mathbf{k}$
5. Now form **F.dl** from 2. and 4. $\left(F_1\frac{df_1}{dt} + F_2\frac{df_2}{dt} + F_3\frac{df_3}{dt}\right)dt$
6. Determine the values of t at the ends of the path: t_A, t_B
7. Set up the integral: $\int_A^B \mathbf{F.dl} = \int_{t_A}^{t_B}\left(F_1\frac{df_1}{dt} + F_2\frac{df_2}{dt} + F_3\frac{df_3}{dt}\right)dt$
8. Evaluate the integral. It should now be in the form of an ordinary integral in terms of the single variable, t.

Example

If $\mathbf{F}(x,y,z) = (3x^2 + 6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, evaluate $\int_C \mathbf{F.dl}$ from (0, 0, 0) to (1, 1, 1):

a) along the path $x = t, y = t^2, z = t^3$

b) along the straight line joining the points.

Solution: a) If $x = t, y = t^2, z = t^3$, then $\mathbf{l}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$

and $\mathbf{dl} = (\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k})\,dt$

If $\mathbf{F}(x,y,z) = (3x^2 + 6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$

then field on path is: $\mathbf{F}(t) = (3t^2 + 6t^2)\mathbf{i} - 14t^5\,\mathbf{j} + 20t^7\,\mathbf{k}$

The end points of the path: when $t = 0$, $\mathbf{l}(t) = (0, 0, 0)$ and when $t = 1$, $\mathbf{l}(t) = (1, 1, 1)$

Set up the integral: $\int_C \mathbf{F.dl} = \int_0^1 \left(9t^2 - 28t^6 + 60t^9\right)dt = \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1 = 5$

b) The straight line joining (0, 0, 0) to (1, 1, 1) is $x = t, y = t, z = t$

$\therefore\ \mathbf{l}(t) = t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}$ so that $\mathbf{dl} = (\mathbf{i}+\mathbf{j}+\mathbf{k})\,dt$ also $\mathbf{F}(t) = (3t^2+6t)\,\mathbf{i} - 14t^2\,\mathbf{j} + 20t^3\,\mathbf{k}$

$$\therefore \int_C \mathbf{F.dl} = \int_0^1 \left(6t - 11t^2 + 20t^3\right)dt = \left[3t^2 - \frac{11}{3}t^3 + 5t^4\right]_0^1 = \frac{13}{3}$$

The value of the integral clearly depends upon the path, so the field **F** cannot be conservative.

Supplementary example

For the force field $\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$, calculate the work done in moving a particle from (1, 0, 0) to (-1, 0, π) along the following paths:

a) the helix $x = \cos\theta$, $y = \sin\theta$, $z = \theta$.
b) the straight line joining the points.

Solution: a) If $x = \cos\theta$, $y = \sin\theta$, $z = \theta$, then $\mathbf{l}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} + \theta\,\mathbf{k}$
and $\mathbf{dl} = (-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} + \mathbf{k})\,d\theta$

Field on path: If $\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ then $\mathbf{F}(\theta) = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} + \theta\,\mathbf{k}$

Work done: $\mathbf{F.dl} = (\sin^2\theta + \cos^2\theta + \theta)\,d\theta = (1 + \theta)\,d\theta$

Ends of path: at (1, 0, 0) $\theta = 0$ at (-1, 0, π) $\theta = \pi$

Line integral: $\int \mathbf{F.dl} = \int_0^{\pi}(1+\theta)d\theta = \left[\theta + \frac{\theta^2}{2}\right]_0^{\pi} = \pi\left(1 + \frac{\pi}{2}\right)$

b) The straight line through the points **a** and **b** is: $\mathbf{l}(\lambda) = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

Therefore, the line through (1,0,0) and (-1,0,π) is: $\mathbf{l}(\lambda) = (1,0,0) + \lambda(-2,0,\pi) = (1-2\lambda)\mathbf{i} + \lambda\pi\mathbf{k}$

so that $\mathbf{dl} = (-2\,\mathbf{i} + \pi\,\mathbf{k})\,d\lambda$

The expression for $\mathbf{l}(\lambda)$ gives x, y and z values of all the points on the path, so we can obtain an expression for the field on the path: $\mathbf{F}(\lambda) = (1 - 2\lambda)\,\mathbf{j} + \lambda\,\pi\,\mathbf{k}$

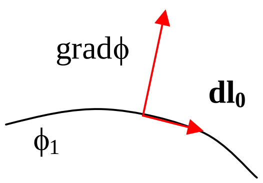
Work done: $\mathbf{F.dl} = \lambda\,\pi^2\,d\lambda$

Ends of path: at (1, 0, 0) $\lambda = 0$ at (-1, 0, π) $\lambda = 1$

Line integral: $\int \mathbf{F.dl} = \int_0^1 \lambda\pi^2\,d\lambda = \pi^2\left[\frac{\lambda^2}{2}\right]_0^1 = \frac{\pi^2}{2}$

Level surfaces

Consider a scalar field ϕ without discontinuities. Connect all points in the region having the same value of ϕ. This will form a surface known as a **level surface**. It is the three-dimensional equivalent of a contour line in two-dimensions.



Let $\mathbf{dl_0}$ lie within a level surface, then the difference in the value of ϕ, $d\phi$, between the ends of the vector $\mathbf{dl_0}$ is zero.

But we have already $d\phi = \text{grad}\phi\,.\,\mathbf{dl}$

so that $0 = \text{grad}\phi\,.\,\mathbf{dl_0}$

In general, neither gradϕ nor $\mathbf{dl_0}$ will be the null vector, so the dot product requires gradϕ and $\mathbf{dl_0}$ to be perpendicular, i.e. **gradϕ is normal to the level surface**.

Mathematical consequence

$\phi(x, y, z) = constant$ defines a surface in three-dimensional space – a level surface.

$\nabla\phi$ is always normal to this surface.

Physical consequence

Consider an equipotential surface $V(x, y, z) = V_0$ in an electric field **E**. Since **E** = – grad$V$, the field vector must be normal to the equipotential surface.

The surface of a static liquid is a surface of equal gravitational potential. The direction of the gravitational field is therefore normal to this surface.

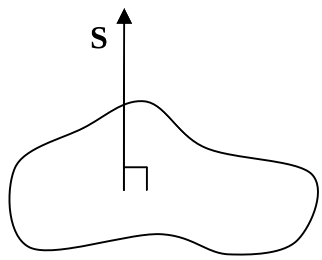
The steepest way uphill is perpendicular to the contours.

In a heated body, level surfaces of temperature are called isotherms. The direction of heat flow is always normal to the isotherms.

Vector area

A concept which is essential to the understanding of the properties of fields is that of **vector area**. It is not immediately obvious that areas can be represented mathematically as vectors, but it is easy to demonstrate. We start with the definition.

If the vector **S** represents a plane area then:



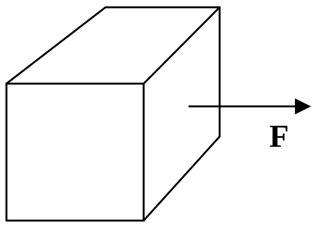
The magnitude of the vector gives the size of the area.

The direction of the vector is normal to the plane of the area.

The vector contains no information about the *shape* of the area.

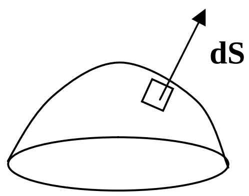
To demonstrate that an area can behave as a vector quantity, consider a hollow cube filled with a gas at a pressure P. Let the area of each face of the cube be A. The force on each face due to the internal pressure will therefore be given by:

$$F = P\,A$$



However, force is a vector quantity and the direction of the force will be normal to the face. The quantity PA must therefore also be a vector normal to the face of the cube. Since pressure is a scalar, we must conclude that A is a vector parallel to F, so we can write:

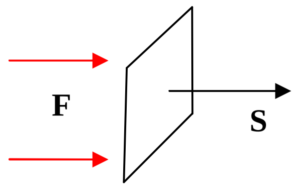
$$\mathbf{F} = P\,\mathbf{A}$$



If the area is on a curved surface, we can take part of the area small enough to be regarded as a plane and represent the small area by the infinitesimal vector **dS**.

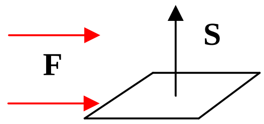
Flux

An important measure of a vector field is its **flux**. Consider a fluid moving with a velocity given by the vector field $\mathbf{F}(x, y, z)$. Its rate of flow (volume / time) may be obtained by observing its velocity through an area represented by the vector **S**:



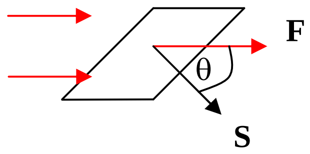
$$\textit{flow rate} = \textit{speed} \times \textit{area} = |\mathbf{F}| \times |\mathbf{S}|$$

If the area is reoriented so that **F** and **S** are orthogonal, no fluid will pass through the area, so the measured flow rate will be zero:



$$\textit{flow rate} = 0$$

A more general case is afforded by orienting the area so there is an angle θ between **F** and **S**. The area presented to the flow is now $|\mathbf{S}| \cos\theta$:



$$\textit{flow rate} = |\mathbf{F}| \times |\mathbf{S}| \cos\theta$$

All of these results can be obtained as the dot product of the two vectors involved:

$$\text{i.e. } \textit{flow rate} = \mathbf{F} \,.\, \mathbf{S}$$

Not all vector fields correspond to a flowing substance (very few do), yet the same measurement can be made on any vector field. Instead of calling it flow rate (because nothing may be flowing) a generic term is used, i.e. **flux**. Flux is simply the Latin for flow. We can therefore refer to the **flux** of an electric or gravitational field and measure it as **F.S**.

It is more useful to measure flux at a point in a field and this is done using an infinitesimal area **dS**, i.e. flux at a point = **F . dS**.

Calculation of **dS** for any surface

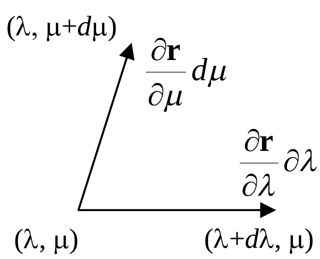
Calculations of flux require a value of **dS**, the element of vector area. The easiest way of achieving this is to express the equation of the surface in parametric form such that $\mathbf{r}(\lambda, \mu)$ gives the position vector of a point on the surface for any λ, μ.

The vector $(\lambda, \mu) \rightarrow (\lambda+d\lambda, \mu)$ lies in the surface and is $\dfrac{\partial \mathbf{r}}{\partial \lambda} d\lambda$

The vector $(\lambda, \mu) \rightarrow (\lambda, \mu+d\mu)$ lies in the surface and is $\dfrac{\partial \mathbf{r}}{\partial \mu} d\mu$

These vectors define an element of area given by

$$\mathbf{dS} = \frac{\partial \mathbf{r}}{\partial \lambda} \times \frac{\partial \mathbf{r}}{\partial \mu} d\lambda \, d\mu$$



Remember that the magnitude of a cross product gives the area of the parallelogram defined by the two vectors and that the direction of the cross product is normal to the plane of the vectors.

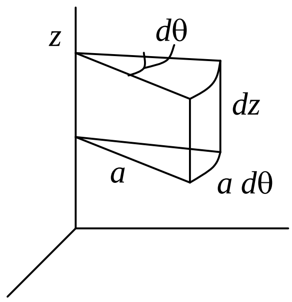
Example – cylindrical surface

Cylindrical polar coordinates are defined by: $x = r\cos\theta$; $y = r\sin\theta$; $z = z$.
A point on a cylinder of radius a is therefore: $x = a\cos\theta$; $y = a\sin\theta$; $z = z$.

This makes the position vector of the point: $\mathbf{r}(\theta, z) = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j} + z\,\mathbf{k}$

$$\therefore \mathbf{dS} = \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z}\, d\theta\, dz = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin\theta & a\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} d\theta\, dz$$

$$= \left(a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j}\right) d\theta\, dz = a\left(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\right) d\theta\, dz$$



This is a unit vector normal to the surface multiplied by the scalar $a\, d\theta\, dz$. This is seen to be the magnitude of **dS** from the diagram.

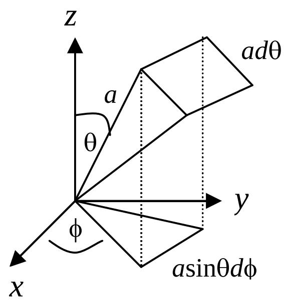
Example – spherical surface

Spherical polar coordinates are defined by: $x = r\cos\phi\sin\theta$; $y = r\sin\phi\sin\theta$; $z = r\cos\theta$
A point on a sphere of radius a is therefore: $x = a\cos\phi\sin\theta$; $y = a\sin\phi\sin\theta$; $z = a\cos\theta$

This makes the position vector of the point: $\mathbf{r}(\theta, \phi) = a\sin\theta\cos\phi\,\mathbf{i} + a\sin\theta\sin\theta\,\mathbf{j} + a\cos\theta\,\mathbf{k}$

$$\therefore \mathbf{dS} = \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi}\, d\theta\, d\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta \\ -a\sin\theta\sin\phi & a\sin\theta\cos\phi & 0 \end{vmatrix} d\theta\, d\phi$$

$$= a^2\sin\theta\left(\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}\right) d\theta\, d\phi$$

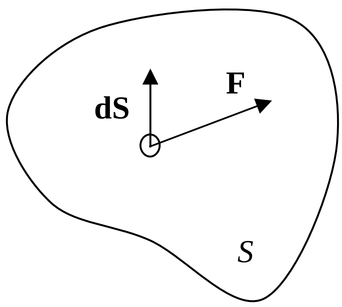


This is a unit vector normal to the spherical surface multiplied by $a^2 \sin\theta\, d\theta\, d\phi$, which is seen to be the magnitude of **dS** from the diagram.

Surface integral

The calculation of flux normally involves the use of a surface integral.

Consider a surface S in a vector field $\mathbf{F}(x, y, z)$. Let **dS** be an element of area on S such that **dS** is always normal to the surface. By convention, **dS** points outwards if S is a closed surface.



The flux of **F** through **dS** is **F . dS**

Therefore, total flux through $S = \iint_S \mathbf{F} \,.\, \mathbf{dS}$

where the integral is taken over the whole surface S. This is a **surface integral**. Note that it is a double integral.

Since $\mathbf{dS} = \mathbf{n}\, dS$ where $\mathbf{n}$ is a unit vector, the surface integral can be written as $\iint_S \mathbf{F} \,.\, \mathbf{n}\, dS$

which is the integral of the normal component of **F** over the surface.

A surface integral may sometimes be written as $\int\limits_S \mathbf{F}\,.\,\mathbf{dS}$ but it is still a double integral.

If the integral is over a closed surface, it may be written as $\oiint\limits_S \mathbf{F}\,.\,\mathbf{dS}$ or $\oint\limits_S \mathbf{F}\,.\,\mathbf{dS}$

Evaluation of a surface integral

The evaluation of a surface integral is carried out in the same way as a line integral, except there are now two variables instead of one.

Example: Evaluate the surface integral $\iint\limits_S \mathbf{F}\,.\,\mathbf{dS}$ where $\mathbf{F}(x,y,z) = z\,\mathbf{i} + x\,\mathbf{j} - 3y^2 z\,\mathbf{k}$ and S is the curved surface of the cylinder $x^2 + y^2 = 16$ in the first octant between $z = 0$ and $z = 2$.

Solution: The equation of the curved surface of a cylinder is $\mathbf{r}(\theta, z) = (4\cos\theta, 4\sin\theta, z)$

From previous results, $\mathbf{dS} = \dfrac{\partial \mathbf{r}}{\partial \theta} \times \dfrac{\partial \mathbf{r}}{\partial z}\, d\theta\, \partial z = 4(\cos\theta, \sin\theta, 0)\, d\theta\, dz$

On the surface of the cylinder, $x = 4\cos\theta$ and $y = 4\sin\theta$. Substitute these into the expression for $\mathbf{F}(x, y, z)$ to obtain the value of $\mathbf{F}$ on the surface:

$$\therefore\ \mathbf{F}(\theta, z) = z\,\mathbf{i} + 4\cos\theta\,\mathbf{j} - 3(4\sin\theta)^2\, z\,\mathbf{k}$$

Now form $\mathbf{F}\,.\,\mathbf{dS}$: $\quad \mathbf{F}\,.\,\mathbf{dS} = 4\big(z\cos\theta + 4\cos\theta\sin\theta\big)\,d\theta\, dz$

The surface integral is therefore $\displaystyle\iint\limits_S \mathbf{F}\,.\,\mathbf{dS} = 4\int_{\theta=0}^{\pi/2}\int_{z=0}^{2}\big(z\cos\theta + 4\sin\theta\cos\theta\big)\,d\theta\, dz$

$$= 4\int_0^{\pi/2}\cos\theta\, d\theta \int_0^2 z\, dz + 8\int_0^{\pi/2}\sin 2\theta\, d\theta \int_0^2 dz$$

$$= 4\big[\sin\theta\big]_0^{\pi/2}\left[\frac{1}{2}z^2\right]_0^2 + 8\left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2}\big[z\big]_0^2 = 4.1.2 + 8.\left(\frac{1}{2} + \frac{1}{2}\right).2 = 24$$

Supplementary example: Evaluate the surface integral $\iint\limits_S \mathbf{F}\,.\,\mathbf{dS}$ where $\mathbf{F}(x,y,z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ and S is the hemispherical surface $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

Solution: The equation of a spherical surface of radius a, centred on the origin is

$$\mathbf{r}(\theta, \phi) = a\ (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

From previous results, $\mathbf{dS} = \dfrac{\partial \mathbf{r}}{\partial \theta} \times \dfrac{\partial \mathbf{r}}{\partial \phi}\, d\theta\, d\phi = a^2 \sin\theta(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,d\theta\, d\phi$

From $\mathbf{r}(\theta,\phi)$ with $a = 2$, $x = 2\sin\theta\cos\phi$, $y = 2\sin\theta\sin\phi$, $z = 2\cos\theta$. Substitute these into the expression for $\mathbf{F}(x,y,z)$ to obtain the value of $\mathbf{F}$ on the surface:

$$\mathbf{F}(\theta, \phi) = (2\sin\theta\sin\phi, 2\sin\theta\cos\phi, 2\cos\theta)$$

Measure of flux: $\quad \mathbf{F.dS} = 4\sin\theta\,(2\sin^2\theta\sin\phi\cos\phi + 2\sin^2\theta\sin\phi\cos\phi + 2\cos^2\theta)\, d\theta\, d\phi$

$$= 8\sin\theta\,(\sin^2\theta\sin 2\phi + \cos^2\theta)\, d\theta\, d\phi$$

Surface integral: $\iint_S \mathbf{F}.\mathbf{dS} = 8\int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \sin\theta\,(\sin^2\theta \sin 2\phi + \cos^2\theta)\,d\theta\,d\phi$

$$= 8\int_0^{\pi/2} \sin^3\theta\,d\theta \int_0^{2\pi} \sin 2\phi\,d\phi + 8\int_0^{\pi/2} \sin\theta \cos^2\theta\,d\theta \int_0^{2\pi} d\phi$$

The first double integral is zero – integration of a whole number of sin waves over 2π.

The second double integral is evaluated using the substitution $u = \cos\theta$ so that $du = -\sin\theta\,d\theta$

$$\therefore \iint_S \mathbf{F}.\mathbf{dS} = -8\int_1^0 u^2\,du \int_0^{2\pi} d\phi = 8 \times \frac{1}{3} \times 2\pi = \frac{16}{3}\pi$$

Volume integral

After line and surface integrals come volume integrals. However, these are just ordinary triple integrals and don't need any special tricks for their evaluation. An example of a volume integral is $\iiint_V \phi\,dV$ where the field of integration is a volume.

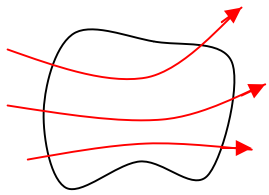
If ϕ is charge density, then ϕdV is the charge contained in the volume dV. The integral will therefore measure the total charge in the volume V.

The concept of divergence of a vector field

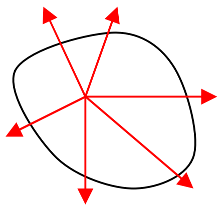
The second important characteristic after the gradient of a scalar field is the divergence of a vector field.

Enclose a volume V in the vector field $\mathbf{F}(x, y, z)$ by an arbitrary closed surface S. The total outward flux of $\mathbf{F}$ through S is given by $\oiint_S \mathbf{F}.\mathbf{dS}$. Now consider the following:

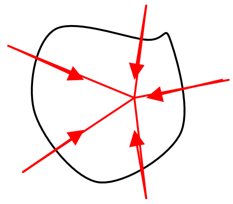
The field inside S is continuous, i.e. there are no sources or sinks. What goes in also comes out, so the inward flux is equal to the outward flux making $\oiint_S \mathbf{F}.\mathbf{dS} = 0$



A source of the field inside S gives a net outward flux through the surface, making $\oiint_S \mathbf{F}.\mathbf{dS} > 0$



There is a sink of the field (a sink is the opposite of a source) inside S. This gives a net inward flux through the surface so that $\oiint_S \mathbf{F}.\mathbf{dS} < 0$



It is clear that the surface integral can be used as a detector of sources or sinks of the field. It gives a measure of the total strength of sources (i.e. sources – sinks) inside S.

For distributed sources, a more useful measure is the source strength per unit volume given by $\frac{1}{V}\oiint_S \mathbf{F}.\mathbf{dS}$ where V is the volume enclosed by S.

It is even more useful to *pin-point* the sources and sinks. This is done by shrinking the volume V to the point where the surface S encloses an infinitesimal volume dV. This will give a measure of flux / unit volume diverging from a point. We therefore define the divergence of the field as

$$\boxed{\text{div}\mathbf{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_S \mathbf{F} \, . \, \mathbf{dS} \quad \text{where } \Delta V \text{ is the volume enclosed by the surface } S.}$$

The limit is finite and independent of the shape of the volume element ΔV.

Note that div**F** defines a **scalar field**.

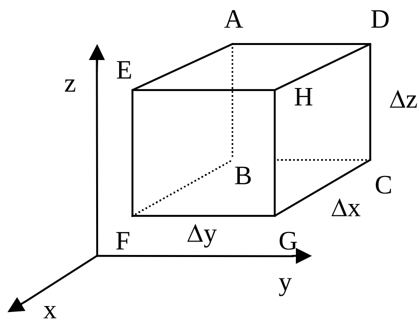
An expression for the divergence of a vector field

Consider the elementary rectangular volume ABCDEFGH whose edges are parallel to the coordinate axes and of lengths Δx, Δy, Δz.

The volume is centred on the point (x, y, z) in the vector field $\mathbf{F}(x, y, z) = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$

Assume that Δx, Δy and Δz are small enough to approximate the value of the field **F** over each face of the box by the value at the mid point of the face.

Now evaluate the surface integral $\oiint_S \mathbf{F} \, . \, \mathbf{dS}$ over the surface of the box by calculating the flux through each face separately.



For the face EFGH: $\quad \mathbf{F} \, . \, \mathbf{dS} = F_1 \, \Delta y \, \Delta z + \frac{\partial}{\partial x}\left(F_1 \, \Delta y \, \Delta z\right)\left(\frac{\Delta x}{2}\right)$

This comes from: **F.dS** at the mid plane + change of **F.dS** upon moving to the front face

For the face ABCD: $\quad \mathbf{F} \, . \, \mathbf{dS} = -F_1 \, \Delta y \, \Delta z + \frac{\partial}{\partial x}\left(-F_1 \, \Delta y \, \Delta z\right)\left(-\frac{\Delta x}{2}\right)$

Adding these gives the total outward flux in the x-direction as $\quad \frac{\partial F_1}{\partial x} \Delta x \, \Delta y \, \Delta z$

Similarly, the total outward flux in the y-direction is $\quad \frac{\partial F_2}{\partial y} \Delta x \, \Delta y \, \Delta z$

and in the z-direction it is $\quad \frac{\partial F_3}{\partial z} \Delta x \, \Delta y \, \Delta z$

The value of the surface integral over the whole surface is the sum of these:

$$\therefore \oiint_S \mathbf{F} \, . \, \mathbf{dS} = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \Delta x \, \Delta y \, \Delta z$$

The volume of the box $= \Delta V = \Delta x \, \Delta y \, \Delta z$

$$\boxed{\therefore \text{div}\mathbf{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_S \mathbf{F} \, . \, \mathbf{dS} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

Divergence can be expressed in terms of the vector differential operator ∇ . Consider:

$$\nabla \cdot \mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) \cdot \left(F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k} \right) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \mathrm{div}\mathbf{F}$$

Mathematical examples

Refer back to the vector field $\mathbf{V}(x, y) = x\,\mathbf{i} + y\,\mathbf{j}$ that was sketched in the first lecture. Imagine that it represents the velocity of flowing water. The flow rate near the origin is small, but the vectors further away from the origin show that the flow rate must increase as distance from the origin increases. Where does the extra water come from? The divergence of the field detects the sources and sinks of the field, so a calculation of the divergence should answer this question:

$$\mathrm{div}\mathbf{V} = \nabla \cdot \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} \right) \cdot \left(x\,\mathbf{i} + y\,\mathbf{j} \right) = 1 + 1 = 2$$

There is positive divergence everywhere (independent of x and y) so there must be a source of water of the same strength over the whole field to produce this particular flow pattern.

A more adventurous example of the calculation of divergence is:

$$\mathrm{div}\left(x\,y\,z\,\mathbf{i} + x\,y^2\,\mathbf{j} + 2\,y\,z^2\,\mathbf{k} \right) = \nabla \cdot \left(x\,y\,z\,\mathbf{i} + x\,y^2\,\mathbf{j} + 2\,y\,z^2\,\mathbf{k} \right) = y\,z + 2\,x\,y + 4\,y\,z = 2\,x\,y + 5\,y\,z$$

The normal rules of calculus apply when manipulating expressions involving ∇ :

$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$ - the derivative of a sum is the sum of derivatives.

$\nabla \cdot (\phi\,\mathbf{A}) = (\nabla \phi) \cdot \mathbf{A} + \phi (\nabla \cdot \mathbf{A})$ - the product rule. Since divergence is a scalar quantity, the terms on the right hand side must also be scalars and this dictates where the dot operators should be.

Before we consider applications of divergence to physics, we must first look at a very important theorem.

The divergence theorem

Consider a finite volume V bounded by a simple closed surface S in a vector field $\mathbf{F}$.

$$\text{Total flux through the surface} = \oint\!\!\oint_S \mathbf{F} \cdot \mathbf{dS}$$

Now, from the definition of divergence: $\mathrm{div}\mathbf{F} = \dfrac{1}{dV} \oint\!\!\oint_{S'} \mathbf{F} \cdot \mathbf{dS'}$ where S' bounds dV

the flux diverging from the volume element dV = $\mathrm{div}\mathbf{F}\,dV$

Therefore, the total flux diverging from V = $\iiint_V \mathrm{div}\mathbf{F}\,dV$

However, flux diverging from V = flux through S

$$\boxed{\therefore \iiint_V \mathrm{div}\mathbf{F}\,dV = \oint\!\!\oint_S \mathbf{F} \cdot \mathbf{dS} \quad \text{where the surface } S \text{ is the boundary of the volume } V}$$

This is the divergence theorem, which can be expressed in words as:

The surface integral of the normal component of a vector field **F** taken over a simple closed surface S is equal to the volume integral of the divergence of **F** taken over the volume bounded by S.

Definition: A simple closed surface can be deformed continuously into a sphere without intersecting itself. Therefore, a torus (the shape of a ring) is not a simple closed surface.

Comment: Mathematically, this relationship between a volume and a surface integral can be used to evaluate either by converting one to the other. However, its importance in physics is immense as we shall soon see.

Electric field

Gauss's law of electrostatics states that:

Total electric flux through a closed surface	=	Total electric charge enclosed by the surface

$$\therefore \oiint_S \mathbf{E}\,.\,\mathbf{dS} = \frac{1}{\varepsilon_0}\iiint_V \rho\, dV$$

where ρ = charge density (coulombs / metre^3)
and ε_0 = permittivity of free space = 8.85×10^{-12} farad / metre

From the divergence theorem: $\oiint_S \mathbf{E}\,.\,\mathbf{dS} = \iiint_V \mathrm{div}\mathbf{E}\, dV$

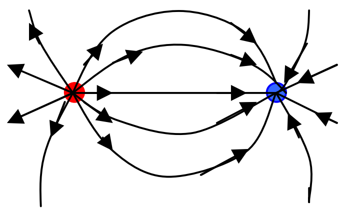
Equate the two volume integrals: $\iiint_V \mathrm{div}\mathbf{E}\, dV = \frac{1}{\varepsilon_0}\iiint_V \rho\, dV$

Since the volume is arbitrary, the integrands must be equal, so we have:

$$\boxed{\mathrm{div}\mathbf{E} = \rho/\varepsilon_0}$$

and this is the **differential form of Gauss's law**.

Since divergence detects and locates the sources and sinks of a field, we see from Gauss's law that the sources of an electric field are electric charges. These are the places where the field lines begin and end, as seen in the sketch of a very simple electric field. Divergence is positive at positive charges and negative at negative charges. Elsewhere, where the field lines are continuous, the divergence is zero.



It is instructive to examine the dimensions of the three integrals in the above analysis:

$$\frac{1}{\varepsilon_0}\iiint_V \rho\, dV = \iiint_V \mathrm{div}\mathbf{E}\, dV = \oiint_S \mathbf{E}\,.\,\mathbf{dS}$$

Their dimensions must all be the same to make physical sense. Let [*x*] mean the dimensions of *x*, then we have:

1st integral: $[\rho][dV]\left[\frac{1}{\varepsilon_0}\right] = \frac{coulomb}{metre^3}metre^3\,\frac{metre}{farad} = \frac{coulomb \times metre}{farad} =$ **volt metres**

Note that $farads = \frac{coulombs}{volts}$, which comes from the formula for capacitance $C = \frac{Q}{V}$

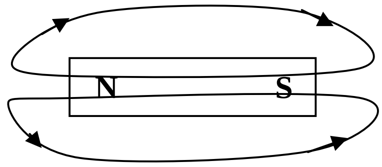
2nd integral: $\left[\mathrm{div}\mathbf{E}\right]\left[dV\right] = \dfrac{volts}{metre^2} metre^3 = \textbf{volt metres}$

Note that [**E**] = V / m, but since $\mathrm{div}\mathbf{E} = \dfrac{\partial E_1}{\partial x} + \dfrac{\partial E_2}{\partial y} + \dfrac{\partial E_3}{\partial z}$, [div**E**] = V / m^2.

3rd integral: $\left[\mathbf{E}\right]\left[\mathbf{dS}\right] = \dfrac{volts}{metres} metres^2 = \textbf{volt metres}$

Magnetic field

In regions where single magnetic poles do not exist (which is everywhere) the divergence of a magnetic field, **H**, must be zero. We can therefore write div**H** = 0, i.e. the field lines are continuous and do not begin or end anywhere. This can be seen in the magnetic field in the diagram.



Bar magnet

Any vector field for which div**F** = 0 everywhere is called **solenoidal**.

[Electric fields are **conservative** whereas magnetic fields are **solenoidal**.]

The Laplace operator

We have seen that an electric field, **E**, is given by **E** = -grad$V$

From Gauss's law we also have $\mathrm{div}\mathbf{E} = \dfrac{\rho}{\varepsilon_0}$

Eliminate **E** to give $\mathrm{div}\,\mathrm{grad}V = -\dfrac{\rho}{\varepsilon_0}$

This result requires that the scalar field V be differentiated twice. The differential operator is:

$$\mathrm{div}\,\mathrm{grad} = \nabla.\nabla = \nabla^2 = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and this is the Laplace operator, usually written as ∇^2. The above equation may now be rewritten as $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$ which is known as **Poisson's equation**.

In regions where there are no free charges, the equation is reduced to

$\boxed{\nabla^2 V = 0}$ which is **Laplace's equation**.

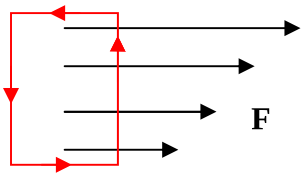
Laplace's equation is one of the most fundamental equations in mathematical physics. It is a second order partial differential equation, having applications in electrostatics, magnetostatics, hydrodynamics, heat flow and many other fields. Its solution is the scalar field $V(x, y, z)$. You will learn how to solve it in a subsequent maths course.

The concept of curl of a vector field

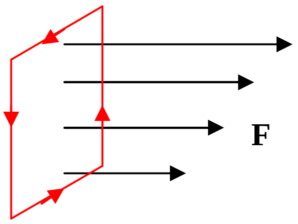
The third of the important field characteristics (after grad and div) that we need to know is called **curl**.

Recall that the value of the line integral round any closed loop in a conservative field is always zero, i.e. $\oint \mathbf{F}.\mathbf{dl} = 0 \quad \text{if} \quad \mathbf{F} = -\operatorname{grad}\phi$

A zero value for this integral is rather special, indicating that the field is special, i.e. it is conservative. In general, this integral will not vanish and its value measures an important property of the vector field. Consider the following.



Take a plane loop in a vector field, **F**, as shown. To keep things easy, let the direction of **F** be everywhere the same, but allow the magnitude to vary as in the diagram. We will now evaluate the line integral of **F.dl** around the loop. With the normal to the loop at right angles to the field, the value of **F.dl** will be zero on the vertical sides, large and negative on the top side and small and positive on the bottom side. The sum of these is certainly not zero.



In the second diagram, the loop has been reoriented so its normal is now parallel to **F**. The value of **F.dl** on all four sides of the loop is now zero. Clearly the value of the line integral depends upon the orientation of the loop. This makes it a function of the vector **n**, the normal to the loop, so it has the property of a vector.

If **F** is a force field, the quantity measured by the line integral is work done by the field. However, the same measure can be made on any vector field, whether it is a force field or not. The generic name for the quantity measured is **circulation**. It is more meaningful to refer circulation to the area of the loop and circulation per unit area is the **curl** of the field. We measure circulation at a point by shrinking the loop to a point and define the curl of the field by the formula:

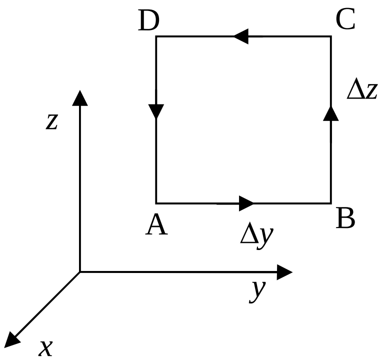
$$\boxed{\mathbf{n}.\operatorname{curl}\mathbf{F} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C \mathbf{F}.\mathbf{dl} \quad \text{where the curve } C \text{ is the boundary of the area } \Delta S}$$

This expression gives the component of curl**F** in the direction of the unit vector **n** which is normal to the plane of the curve *C*. Clearly, curl**F** is a vector field.

Sign convention: the direction of circulation round the loop and the direction of the normal to the loop form a right-handed screw.

The curl of a vector field will be non-zero wherever the field possesses shear (as in the diagrams above) and certain kinds of rotation. We will examine this more closely when we consider applications in physics.

Curl of a vector field in Cartesian coordinates



We will derive an expression for the curl of a vector field by evaluating the line integral $\frac{1}{\Delta y\,\Delta z}\oint_C \mathbf{F}.\mathbf{dl}$ round the closed loop ABCD in the vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. The sides of the loop are parallel to the coordinate axes and are of lengths Δy and Δz.

The sign convention means that for the direction of circulation shown, the integral will evaluate the positive *x*-

component of curl**F**.

The centre of the elementary loop ABCD is at (x, y, z).

Assume Δy and Δz are both small enough so the value of the vector field along each line can be approximated by its value at the mid point of the line.

Now $\oint \mathbf{F}\,.\mathbf{dl} = \int_A^B \mathbf{F}\,.\mathbf{dl} + \int_B^C \mathbf{F}\,.\mathbf{dl} + \int_C^D \mathbf{F}\,.\mathbf{dl} + \int_D^A \mathbf{F}\,.\mathbf{dl}$

$$= \left(F_2\,\Delta y + \frac{\partial}{\partial z}\left(F_2\,\Delta y\right)\left(-\frac{\Delta z}{2}\right)\right) + \left(F_3\,\Delta z + \frac{\partial}{\partial y}\left(F_3\,\Delta z\right)\left(\frac{\Delta y}{2}\right)\right)$$

$$+\left(-F_2\,\Delta y + \frac{\partial}{\partial z}\left(-F_2\,\Delta y\right)\left(\frac{\Delta z}{2}\right)\right) + \left(-F_3\,\Delta z + \frac{\partial}{\partial y}\left(-F_3\,\Delta z\right)\left(-\frac{\Delta y}{2}\right)\right)$$

$$= \frac{\partial}{\partial y}\left(F_3\,\Delta z\right)\Delta y - \frac{\partial}{\partial z}\left(F_2\,\Delta y\right)\Delta z \qquad = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\Delta y\,\Delta z$$

$\therefore$ the x-component of curl**F** $= \dfrac{1}{\Delta y\,\Delta z}\oint \mathbf{F}\,.\mathbf{dl} = \dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}$

This expression can be used as a pattern to obtain the other components of curl**F** giving:

$$\boxed{\text{curl}\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}}$$

Just as grad and div can be expressed using the ∇ operator, curl can too. Consider

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} = \text{curl}\mathbf{F}$$

We now have a complete set of field characteristics:

$$\boxed{\begin{array}{ll} \text{grad}\phi = \nabla\phi & \text{(vector field)} \\ \text{div}\mathbf{F} = \nabla\,.\mathbf{F} & \text{(scalar field)} \\ \text{curl}\mathbf{F} = \nabla\times\mathbf{F} & \text{(vector field)} \end{array}}$$

As before, the normal rules of calculus apply:

$\nabla\times\left(\mathbf{A}+\mathbf{B}\right) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$ - the derivative of a sum is the sum of derivatives.

$\nabla\times\left(\phi\,\mathbf{A}\right) = \nabla\phi\times\mathbf{A} + \phi\nabla\times\mathbf{A}$ - the product rule. Note that the cross operator always results in the right hand side terms being vectors.

Conservative field revisited: Since curl**F** is measured by $\oint \mathbf{F}\,.\mathbf{dl}$ and $\oint \mathbf{F}\,.\mathbf{dl} = 0$ for all closed paths in a conservative field, then a conservative field must have curl**F** = **0** everywhere. This is the easiest test for a conservative field.

Note that we now have:

$$\boxed{\begin{array}{r}\textbf{conservative field} \text{ has } \text{curl}\mathbf{F} = 0\\ \textbf{solenoidal field} \text{ has } \text{div}\mathbf{F} = 0\end{array}}$$

Mathematical example

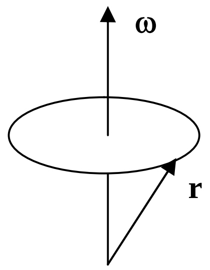
If $\mathbf{A} = x^2y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, find curl curl$\mathbf{A}$.

Solution: Because curl$\mathbf{A}$ is a vector field, its curl can also be determined, so the field is differentiated twice:

$$\text{curl curl}\mathbf{A} = \nabla\times(\nabla\times\mathbf{A}) = \nabla\times\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x^2y & -2xz & 2yz\end{vmatrix} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 2z+2x & 0 & -2z-x^2\end{vmatrix} = (2+2x)\,\mathbf{j}$$

Rigid body rotation

The first of our physical examples examines a rotating field to see what its value of curl is. The system considered is a rotating disc. The linear velocity of each point on the disc constitutes a vector field and the linear velocity is given by



$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$$

where $\boldsymbol{\omega}$ is the angular velocity, which points along the axis of rotation, and $\mathbf{r}$ is the position vector of a point on the disc

Let us now obtain an expression for the curl of $\mathbf{v}$. Define $\boldsymbol{\omega}$ as $(\omega_1, \omega_2, \omega_3)$ and $\mathbf{r}$ as (x, y, z):

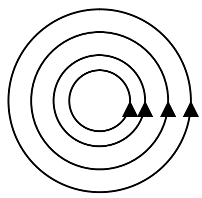
$$\text{curl}\mathbf{v} = \nabla\times(\boldsymbol{\omega}\times\mathbf{r}) = \nabla\times\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \omega_1 & \omega_2 & \omega_3\\ x & y & z\end{vmatrix} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \omega_2 z-\omega_3 y & \omega_3 x-\omega_1 z & \omega_1 y-\omega_2 x\end{vmatrix}$$

$$= (\omega_1+\omega_1)\,\mathbf{i} + (\omega_2+\omega_2)\,\mathbf{j} + (\omega_3+\omega_3)\,\mathbf{k} = 2\boldsymbol{\omega}$$

This shows that the circulation is constant over the whole field $\mathbf{v}$ (its value is independent of position).

Comment: Perhaps it is no surprise that a rotating system has lots of circulation and therefore a non-zero value of curl. However, before jumping to conclusions, let us examine another rotating system in the same way.

The field lines of $\mathbf{v}$ for the rotating disc are concentric circles. A field with virtually the same lines is the velocity of water flowing down a plug-hole. The field lines cannot be exactly concentric circles, but they can be very close, so we will assume they are to simplify the mathematics. It will be instructive to compare the two fields.



Water down a plug-hole

As before, we will determine the curl of the linear velocity field. First of all we need to determine how the water moves - it doesn't behave as a rigid body like the rotating disc.

Consider a small parcel of water of mass m at radius r with angular velocity ω.

Angular momentum about the centre of the field = moment of inertia × angular velocity = Iω

Ignoring viscosity and friction, angular momentum is constant.

$$\therefore I\,\omega = m\,r^2\,\omega = \text{constant} \quad \text{giving} \quad \omega = \frac{\text{constant}}{m\,r^2}$$

This makes the linear velocity $= v = \omega\,r = \dfrac{\text{constant}}{m\,r^2}r = \dfrac{\text{constant}}{m\,r}$ that is, $v \propto \dfrac{1}{r}$

A vector field whose field lines are concentric circles and whose magnitude is $1/r$ is:

$$\mathbf{v} = \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$$

We can now calculate curl**v** as

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix} = \left(\frac{1}{x^2+y^2} - x(x^2+y^2)^{-2}2x + \frac{1}{x^2+y^2} - y(x^2+y^2)^{-2}2y\right)\mathbf{k}$$

$$= \left[\frac{2}{x^2+y^2} - \frac{2x^2+2y^2}{(x^2+y^2)^2}\right]\mathbf{k} = \mathbf{0}$$

Comment: This result of obtaining zero curl for a whirlpool may be surprising if we had tried to predict the result. What is it telling us about the flow of water? What is the essential difference between this and the rotating disc?

If we place a marker on the rotating disc, it will be carried around a field line, i.e. it will interact with the field, changing orientation as it goes. It will complete one rotation for each revolution of the disc. This is obvious.

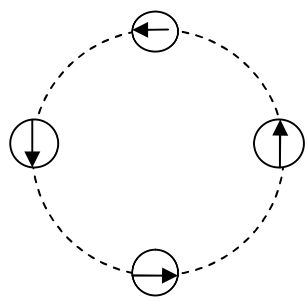
If we float a cork on the whirlpool, it will also interact with the field, but how will its orientation change? This is not so obvious and we really need to do the experiment. The OU video of this (shown as part of the course) shows that the orientation of the cork does not change as it is carried around by the field. This lack of rotation is what is predicted by the zero curl.

Curl is seen to be a measure of that property of a vector field which will change the orientation of an object interacting with the field.

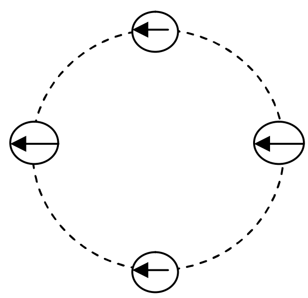
The axis of rotation is the direction of curl.

The speed of rotation is measured by the magnitude of curl.

Just as the concept of divergence gave rise to the very important divergence theorem, there is an equally important theorem involving curl – Stokes's theorem.

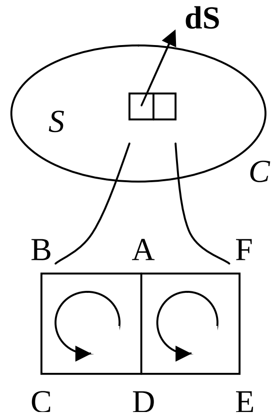


Rigid disc: $v \propto r$



Whirlpool: $v \propto 1/r$

Stokes's theorem



Consider a surface S bounded by a closed curve C in a vector field $\mathbf{F}$.

Consider also an element of area ABCD on S and its adjacent element ADEF, enlarged in the lower diagram.

For ABCD: $\quad \oint_{ABCD} \mathbf{F}\,.\,\mathbf{dl} = \left(\int_A^B + \int_B^C + \int_C^D + \int_D^A \right) \mathbf{F}\,.\,\mathbf{dl}$

and for ADEF: $\quad \oint_{ADEF} \mathbf{F}\,.\,\mathbf{dl} = \left(\int_D^E + \int_E^F + \int_F^A + \int_A^D \right) \mathbf{F}\,.\,\mathbf{dl}$

But $\quad \int_D^A \mathbf{F}\,.\,\mathbf{dl} = -\int_A^D \mathbf{F}\,.\,\mathbf{dl}$

Therefore $\quad \left(\oint_{ABCD} + \oint_{ADEF} \right) \mathbf{F}\,.\,\mathbf{dl} = \left(\int_A^B + \int_B^C + \int_C^D + \int_D^E + \int_E^F + \int_F^A \right) \mathbf{F}\,.\,\mathbf{dl} = \oint_{ABCDEF} \mathbf{F}\,.\,\mathbf{dl}$

which is around the boundary of the combined area.

From the definition of curl: $\quad \mathrm{curl}\mathbf{F}.\mathbf{n} = \frac{1}{dS} \oint \mathbf{F}.\,\mathbf{dl}$

so that $\quad \mathrm{curl}\mathbf{F}\,.\,\mathbf{dS} = \oint \mathbf{F}\,.\,\mathbf{dl}$ where the line integral is round the boundary of **dS.**

Integration of this expression over the whole surface S is achieved by adding up the contributions of adjacent elements of area. As each new element is added, the line integral on the right is always round the boundary of the combined area as seen in the above analysis. The final result is therefore:

$$\boxed{\iint_S \mathrm{curl}\mathbf{F}\,.\,\mathbf{dS} = \oint_C \mathbf{F}\,.\,\mathbf{dl} \text{ where } C \text{ is the boundary of } S.}$$

This is Stokes's theorem which can be expressed in words as:

The line integral of the tangential component of a vector field **F** taken round a simple closed curve C is equal to the surface integral of the normal component of the curl of **F** taken over any surface having C as its boundary.

Definition: A simple closed curve can be continuously reduced to a point without intersecting itself, i.e. it doesn't form a knot.

Comment: Mathematically, this relationship between a surface and a line integral can be used to evaluate either by converting one to the other. However, it is an extremely important theorem of immense use in physics as we shall soon see.

Summary of integral relationships

We now have a set of three quite remarkable integral relationships and it is worth looking at them all together:

Divergence theorem: $\quad \iiint_V \mathrm{div}\mathbf{F}\, dV = \oint\!\!\oint_S \mathbf{F}\,.\,\mathbf{dS}$ where surface S is the boundary of volume V.

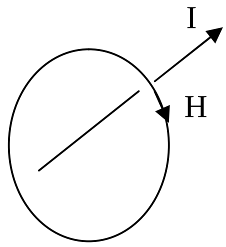
Stokes's theorem: $\iint_S \text{curl}\mathbf{F}\,.\,d\mathbf{S} = \oint_C \mathbf{F}\,.\,d\mathbf{l}$ where curve C is the boundary of surface S.

Conservative field: $\int_A^B \text{grad}\phi\,.\,d\mathbf{l} = \phi_B - \phi_A$ where A and B are the ends of a defined curve.

They involve the three major field characteristics – grad, div and curl – that we have been considering. In each case, the expression on the right involves only the boundary of the field of integration on the left. The left-hand-side integrals can therefore be evaluated without knowing the detailed behaviour of the field inside the boundary.

Ampère's law

Ampère's law for a steady current relates the magnetic field around a current-carrying conductor to the electric current flowing along the conductor:



$$\oint_C \mathbf{H}.d\mathbf{l} = I$$

Circulation of magnetic field round a closed loop = current through loop

Be careful about the physical dimensions in this expression. Note that:

$\mathbf{B} = \mu_0 \mathbf{H}$ in vacuo

$\mathbf{H}$ = magnetising force ampere / metre
$\mathbf{B}$ = magnetic induction weber / metre^2
μ_0 = permeability of free space = $4\pi\times10^{-7}$ henry / metre

From Stokes's theorem: $\oint_C \mathbf{H}.\,d\mathbf{l} = \iint_S \text{curl}\mathbf{H}.\,d\mathbf{S}$ where S is any surface having C as its boundary.

Now express the current in terms of current density **J,** measured in amps / m^2, i.e. current flowing through unit area. The vector direction gives the direction of flow of the current.

We find that: $I = \iint_S \mathbf{J}\,.\,d\mathbf{S}$ where the surface integral is over the same surface S as in Stokes's theorem, i.e. flux of **J** = current.

Substitute for I from Ampère's law and make use of Stokes's theorem to give

$$\therefore \oint_c \mathbf{H}.\,d\mathbf{l} = \iint_S \mathbf{J}.\,d\mathbf{S} \quad \text{so that} \quad \iint_S \text{curl}\mathbf{H}\,.\,d\mathbf{S} = \iint_S \mathbf{J}\,.\,d\mathbf{S}$$

Since the surface S is arbitrary, the integrands must be equal, so that:

$$\boxed{\text{curl}\mathbf{H} = \mathbf{J}}$$

and this is **the differential form of Ampère's law.**

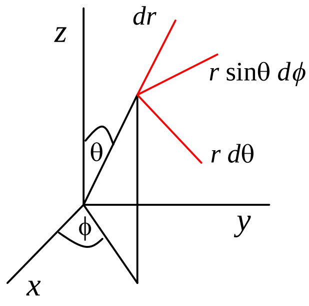
Grad in spherical polar coordinates

So far, we have always used Cartesian coordinates to develop expressions for curl, div, grad and ∇^2. However, it is useful to have the equivalent expressions in cylindrical and spherical polar coordinates as well.

Consider a scalar field $f(r, \theta, \phi)$.

We already know that $df = \text{grad}f \,.\, \mathbf{dl}$

The total differential of f is: $df = \dfrac{\partial f}{\partial r}dr + \dfrac{\partial f}{\partial \theta}d\theta + \dfrac{\partial f}{\partial \phi}d\phi$



From the diagram: $\mathbf{dl} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\varphi}}$

where $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ are dimensionless unit vectors in the directions of increasing r, θ, ϕ respectively, so that

$$\hat{\mathbf{r}}.\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}.\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}}.\hat{\boldsymbol{\varphi}} = 1, \quad \hat{\mathbf{r}}.\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}.\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\theta}}.\hat{\boldsymbol{\varphi}} = 0$$

Let $\text{grad}f = A_r\,\hat{\mathbf{r}} + A_\theta\,\hat{\boldsymbol{\theta}} + A_\phi\,\hat{\boldsymbol{\varphi}}$

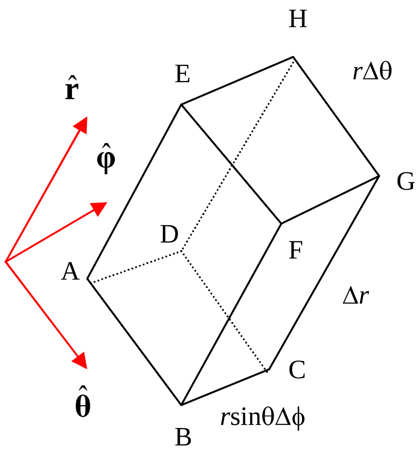
Then $\text{grad}f\,.\mathbf{dl} = \left(A_r\,\hat{\mathbf{r}} + A_\theta\,\hat{\boldsymbol{\theta}} + A_\phi\,\hat{\boldsymbol{\varphi}}\right).\left(dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\varphi}}\right) = A_r\,dr + A_\theta\,r\,d\theta + A_\phi\,r\sin\theta\,d\phi$

Comparing terms in df and $\text{grad}f\,.\,\mathbf{dl}$ gives: $A_r = \dfrac{\partial f}{\partial r}$; $\quad A_\theta = \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}$; $\quad A_\phi = \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \phi}$

$$\boxed{\therefore\ \text{grad}f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\varphi}}}$$

Div in spherical polar coordinates

Consider an elementary volume ABCDEFGH centred around the point (r, θ, ϕ) in the vector field $\mathbf{F}(r,\theta,\phi) = F_r\,\hat{\mathbf{r}} + F_\theta\,\hat{\boldsymbol{\theta}} + F_\phi\,\hat{\boldsymbol{\varphi}}$



Assume that Δr, Δθ, Δϕ are all small enough so the field **F** on each face of ABCDEFGH can be approximated by its value at the mid point.

To determine div**F**, use $\text{div}\mathbf{F} = \displaystyle\lim_{\Delta V \to 0} \frac{1}{\Delta V}\oiint_S \mathbf{F}.\mathbf{dS}$

where S is the surface of ΔV.

Flux through face ABCD (= **F.dS**) $= -F_r\,r^2\sin\theta\,\Delta\theta\,\Delta\phi + \dfrac{\partial}{\partial r}\left(-F_r\,r^2\sin\theta\,\Delta\theta\,\Delta\phi\right)\left(-\dfrac{\Delta r}{2}\right)$

Flux through face EFGH $= F_r\,r^2\sin\theta\,\Delta\theta\,\Delta\phi + \dfrac{\partial}{\partial r}\left(F_r\,r^2\sin\theta\,\Delta\theta\,\Delta\phi\right)\left(\dfrac{\Delta r}{2}\right)$

$\therefore$ total outward flux in $\hat{\mathbf{r}}$-direction $= \dfrac{\partial}{\partial r}\left(F_r\,r^2\sin\theta\,\Delta\theta\,\Delta\phi\right)(\Delta r)$

Similarly, total flux in $\hat{\boldsymbol{\theta}}$-direction $= \dfrac{\partial}{\partial \theta}\left(F_\theta\,r\sin\theta\,\Delta r\,\Delta\phi\right)(\Delta\theta)$

Total flux in the $\hat{\boldsymbol{\varphi}}$-direction $= \dfrac{\partial}{\partial \phi}\left(F_\phi\,r\,\Delta r\,\Delta\theta\right)(\Delta\phi)$

$$\therefore \oiint_S \mathbf{F}.\mathbf{dS} = \left[\sin\theta\frac{\partial}{\partial r}\left(r^2 F_r\right) + r\frac{\partial}{\partial \theta}\left(\sin\theta\,F_\theta\right) + r\frac{\partial}{\partial \phi}\left(F_\phi\right)\right]\Delta r\,\Delta\theta\,\Delta\phi$$

Elementary volume $\Delta V = r^2 \sin\theta \, \Delta r \, \Delta\theta \, \Delta\phi$

$$\boxed{\text{div}\mathbf{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_S \mathbf{F} . \mathbf{dS} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, F_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(F_\phi\right)}$$

The Laplacian operator in spherical polar coordinates

The Laplacian operator, ∇^2, is given by div grad and we have just developed expressions for both grad and div above.

$$\nabla^2 = \text{div grad} = \text{div}\left(\frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\boldsymbol{\varphi}}\right)$$

$$\boxed{\therefore \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}$$

Comment: You are not expected to remember complicated expressions like these. You are given the complete set of expressions for grad, div, curl and ∇^2 in Cartesians, spherical and cylindrical polars in the formula book in exams. However, you are expected to be able to use them. You will come across them again in physics courses.

GRAD, DIV, CURL AND ∇^2 IN DIFFERENT COORDINATE SYSTEMS

Cartesian coordinates

$$\text{grad}\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

$$div\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$curl\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Cylindrical polar coordinates

$$grad(f) = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

$$div(\mathbf{F}) = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\,F_r\right) + \frac{\partial}{\partial \theta}\left(F_\theta\right) + \frac{\partial}{\partial z}\left(rF_z\right)\right]$$

$$curl(\mathbf{F}) = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\,\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ F_r & r\,F_\theta & F_z \end{vmatrix}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Spherical polar coordinates

$$grad(f) = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\varphi}}$$

$$div(\mathbf{F}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\,F_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(F_\phi\right)$$

$$curl(\mathbf{F}) = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\,\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\varphi}} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ F_r & r\,F_\theta & r\sin\theta\,F_\phi \end{vmatrix}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

The continuity equation

We will now bring in *time* as an extra variable. Most things move or change with time and we have to be able to deal with that.

Consider a substance of density $\rho(x, y, z, t)$ with a flow rate (flux) given by $\mathbf{J}(x, y, z, t)$ as amount of substance / unit area / unit time.

Consider also an arbitrary volume V bounded by the closed surface S.

Total amount of substance in V = $\iiint_V \rho \, dV$

$\therefore$ rate of increase of substance in V = $\frac{\partial}{\partial t}\iiint_V \rho \, dV$

Assuming ρ to be continuous in V with respect to space and time, we can reverse the order of operations so that $\frac{\partial}{\partial t}\iiint_V \quad = \iiint_V \frac{\partial}{\partial t}$

$\therefore$ rate of increase of substance in V = $\underline{\iiint_V \frac{\partial \rho}{\partial t} dV}$

Now measure the same quantity in a different way and equate the two results.

Amount of substance flowing through element of area on S in unit time = $\mathbf{J} \cdot \mathbf{dS}$

Because $\mathbf{dS}$ points outwards, this measures amount of substance lost from V.

$\therefore$ rate of decrease of substance from V = $\oiint_S \mathbf{J} \cdot \mathbf{dS}$

But, from the divergence theorem $\oiint_S \mathbf{J} \cdot \mathbf{dS} = \iiint_V \operatorname{div}\mathbf{J} \, dV$

$\therefore$ rate of decrease of substance in V = $\underline{\iiint_V \operatorname{div}\mathbf{J} \, dV}$

If there are no sources or sinks in V, these two results must be the same. We therefore have

$$\iiint_V \frac{\partial \rho}{\partial t} dV = -\iiint_V \operatorname{div}\mathbf{J} \, dV$$

Since V is arbitrary, the integrands must be equal. This gives the continuity equation as:

$$\boxed{\operatorname{div}\mathbf{J} + \frac{\partial \rho}{\partial t} = 0}$$

Examples

1. Fluid (gas or liquid)

$\mathbf{J} = \rho\mathbf{v}$ with dimensions mass /unit area / unit time $\qquad$ $[\operatorname{div}\mathbf{J}]$ = mass / vol / time

where $\mathbf{v}$ = velocity
and ρ = density $\qquad$ $[\partial\rho/\partial t]$ = mass / vol / time

If ρ increases at a point making $\partial\rho/\partial t > 0$, this can only be achieved by a net flow of fluid towards the point making div**J** < 0 such that $\text{div}\mathbf{J} + \partial\rho/\partial t = 0$.

If the fluid is incompressible, then $\partial\rho/\partial t = 0$ so that div**J** = 0. This means that div**v** = 0 and **v** must therefore be a solenoidal field.

2. Electricity

J = current density (coulombs / area / time) [div**J**] = coulombs / vol / time

ρ = charge density (coulombs / vol) $[\partial\rho/\partial t]$ = coulombs / vol / time

A build-up of charge at a point, making $\partial\rho/\partial t > 0$, is only possible by a net flow of charge towards the point, making div**J** < 0 such that $\text{div}\mathbf{J} + \partial\rho/\partial t = 0$.

In a good electrical conductor, there can be no build-up of charge. This means that $\partial\rho/\partial t = 0$ so that div**J** = 0 and **J** is solenoidal.

3. Heat

J = heat flow (quantity of heat / area / time) [div**J**] = heat / vol / time

ρ = heat density = quantity of heat / vol $[\partial\rho/\partial t]$ = heat / vol / time

$$= \frac{\text{mass} \times \text{specific heat} \times \text{temperature}}{\text{vol}}$$

= density of heat conductor × specific heat × temperature

The more usual symbols used for heat are:

Q = heat flow; ρ = density of conductor; σ = specific heat; θ = temperature

Assuming ρ and σ to be constant with time, the continuity equation for heat becomes:

$$\text{div}\mathbf{Q} + \rho\sigma\frac{\partial\theta}{\partial t} = 0$$

The temperature can only decrease $(\partial\theta/\partial t < 0)$ by a net flow of heat away, i.e. div**Q** > 0 such that $\text{div}\mathbf{Q} + \rho\sigma\,\partial\theta/\partial t = 0$.

In steady state heat conduction, the temperature will be constant with time so that $\partial\theta/\partial t = 0$ giving div**Q** = 0 and **Q** must be solenoidal.

Heat conduction equation

We have already seen that heat conduction is described by **Fourier's first law of heat conduction**: **Q** = - k gradθ where k is thermal conductivity

Therefore div**Q** = - k div gradθ = $-k\nabla^2\theta$

But from the continuity equation $\text{div}\mathbf{Q} = -\rho\sigma\frac{\partial\theta}{\partial t}$

Eliminating div**Q** gives the heat conduction equation $\boxed{\nabla^2\theta = \frac{\rho\sigma}{k}\frac{\partial\theta}{\partial t}}$

This is also known as **Fourier's second law of heat conduction.**

The quantity $\dfrac{k}{\rho\,\sigma}$ is known as the **thermal diffusivity.**

For steady state heat conduction, $\partial\theta/\partial t = 0$ then $\boxed{\nabla^2\theta = 0}$

Diffusion equation

The heat conduction equation will also describe the more general process of diffusion, as heat conduction is a particular example of diffusion.

The diffusion coefficient of a substance is defined by:

flow rate = - diffusion coefficient × concentration gradient

This is described by the equation $c\,\mathbf{v} = -D\,\mathrm{grad}c$

where c = concentration (mass / vol) and $\mathbf{v}$ = velocity (m / s)

This makes the dimensions of the diffusion coefficient, D, = area / time

Take the divergence of both sides of the above equation to get:

$$\mathrm{div}(c\,\mathbf{v}) = -D\,\mathrm{div}\,\mathrm{grad}c = -D\nabla^2 c$$

But the continuity equation gives $\mathrm{div}(c\,\mathbf{v}) + \dfrac{\partial c}{\partial t} = 0$

Eliminating div(c**v**) gives the **diffusion equation** $\boxed{\nabla^2 c = \dfrac{1}{D}\dfrac{\partial c}{\partial t}}$

Some partial differential equations of physics

We can now write down and compare three very important partial differential equations in mathematical physics:

Laplace's equation

$\boxed{\nabla^2\phi = 0}$

This equation has applications in electricity, magnetism, gravitation and steady-state heat flow among many others. The scalar field ϕ may be electric potential, gravitational potential, temperature etc., depending upon the application.

Laplace (1749-1827) used it to study the gravitational attraction of extended masses.

Diffusion equation

$\boxed{\nabla^2\phi = \dfrac{1}{D}\dfrac{\partial\phi}{\partial t}}$

The obvious applications are in heat conduction and diffusion.
The diffusion coefficient, D, becomes $k / \rho\sigma$ for heat flow.

The scalar field ϕ may be concentration, temperature, partial pressure, depending upon the application.
Joseph Fourier (1768-1830) developed Fourier series as a means of solving it. We will come across Fourier series later this term.

Wave equation

You will have encountered the wave equation in your waves course in the first year. It describes the behaviour of all waves that propagate at **constant speed** and **constant profile**, i.e. they do not change shape as they travel.

$$\boxed{\nabla^2\phi = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}}$$

The field ϕ may be displacement, pressure, electric field, magnetic field etc. depending upon the nature of the wave (sound wave, electromagnetic wave etc.).
v is the speed of propagation.

Useful relationships

We have already seen a number of vector relationships that give the grad, div or curl of sums and products of fields. A more complete list is available in exams if you need to use them, but they will also be given in exam questions as appropriate. Four more relationships are presented here which are both useful and interesting. They all involve second order derivatives.

1. curl gradϕ = 0 gradϕ produces a conservative field, where ϕ is the **scalar potential** of that field, and the curl of a conservative field is everywhere zero.

2. div curlF = 0 We know that the divergence of a solenoidal field is zero. It is clear, therefore, that curl**F** produces a solenoidal field and **F** is known as the **vector potential** of that field.

Proof of the relationship is accomplished by expanding the field into its Cartesian components, i.e. $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ so that

$$\operatorname{div}\operatorname{curl}\mathbf{F} = \nabla.\nabla\times\mathbf{F} = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x\,\partial y}-\frac{\partial^2 F_2}{\partial x\,\partial z}+\frac{\partial^2 F_1}{\partial y\,\partial z}-\frac{\partial^2 F_3}{\partial y\,\partial x}+\frac{\partial^2 F_2}{\partial z\,\partial x}-\frac{\partial^2 F_1}{\partial z\,\partial y} = 0$$

3. div gradϕ = $\nabla^2\varphi$ This produces the Laplace operator.

4. $\mathbf{curl\,curlF} = \nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla.\mathbf{F})-\nabla^2\mathbf{F} = \mathbf{grad\,divF}-\nabla^2\mathbf{F}$ This is one of the more famous vector relationships as we shall soon see. It will be used in one of the most remarkable pieces of mathematical physics ever and will form a suitable climax to this course.

To prove this relationship, use the standard relationship for dealing with vector triple products, i.e. **a×(b×c) = (a.c) b – (a.b) c**

Let **a** = **b** = ∇ and **c** = **F**. However, since ∇ is an operator, the order of terms on the right-hand-side becomes important. If we rewrite the relationship as **a×(b×c) = b (a.c) – (a.b) c** so the operators come first in each term, we can see that this is exactly the relationship we wish to prove.

Ampère's law revisited

We have already derived the differential form of Ampère's law for steady currents, curl**H** = **J**

This does not allow variations of **J** with time because div curl**H =** div**J**

But div curl**A** = 0 for any differentiable vector field **A**, making div**J** = 0

However, the continuity equation gives $\operatorname{div}\mathbf{J}+\frac{\partial\rho}{\partial t} = 0$ so that $\operatorname{div}\mathbf{J} = -\frac{\partial\rho}{\partial t}$

There is clearly a time-dependent term missing from Ampère's law, but we can find it by making it consistent with the continuity equation.

Substitute for ρ in the continuity equation from Gauss's law div$\mathbf{E} = \rho / \varepsilon_0$, i.e. $\rho = \varepsilon_0$ div$\mathbf{E}$

The continuity equation becomes $\mathrm{div}\mathbf{J} + \frac{\partial}{\partial t}(\varepsilon_0\, \mathrm{div}\mathbf{E}) = 0$

Changing the order of the differential operators gives

$$\mathrm{div}\mathbf{J} + \mathrm{div}\left(\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) = \mathrm{div}\left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) = 0$$

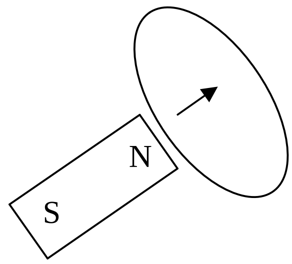
It would appear, then, that $\mathbf{J}$ in Ampère's law should be replaced by $\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ to allow for time variations.

Therefore, Ampère's law becomes $\boxed{\mathrm{curl}\mathbf{H} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}}$

Faraday's law of electromagnetic induction

Faraday's law states that:

voltage round loop = - rate of change of magnetic flux through loop



Expressing this mathematically gives:

$\oint_C \mathbf{E}.\mathbf{dl} = -\frac{\partial}{\partial t}\iint_S \mu_0\, \mathbf{H}.\mathbf{dS}$ where C is the loop in the diagram and S is a surface with C as its boundary.

By Stokes's theorem: $\oint_C \mathbf{E}.\mathbf{dl} = \iint_S \mathrm{curl}\mathbf{E}.\mathbf{dS}$

$$\therefore \iint_S \mathrm{curl}\mathbf{E}.\mathbf{dS} = -\frac{\partial}{\partial t}\iint_S \mu_0\, \mathbf{H}.\mathbf{dS} = -\mu_0 \iint_S \frac{\partial \mathbf{H}}{\partial t}.\mathbf{dS}$$

Since the surface S is arbitrary, the integrands must be equal. This gives the differential form of Faraday's law as:

$$\boxed{\mathrm{curl}\mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}}$$

Just to make sure that the physical dimensions are correct, we see that [$\mathbf{E}$] = V / m so that [curl$\mathbf{E}$] = V / m^2.

Also, μ_0 = permeability of free space = $4\pi \times 10^{-7}$ henrys / metre and henry = volt sec / amp

$\mathbf{H}$ = magnetising force measured in amp / metre so that $\left[\mu_0 \frac{\partial \mathbf{H}}{\partial t}\right] = \frac{V\,s}{Am}\frac{A}{m\,s} = V/m^2$

Note that $\mathbf{B} = \mu_0\, \mathbf{H}$ = magnetic induction measured in weber / metre^2.

Maxwell’s equations

Let us now assemble the various laws of electricity and magnetism that we have encountered in their differential forms:

Gauss’s law of electrostatics $\quad \mathrm{div}\mathbf{E} = \rho / \varepsilon_0$

Gauss’s law of magnetostatics $\quad \mathrm{div}\mathbf{H} = 0$

Faraday’s law $\quad \mathrm{curl}\mathbf{E} = -\mu_0 \dfrac{\partial \mathbf{H}}{\partial t}$

Ampère’s law $\quad \mathrm{curl}\mathbf{H} = \mathbf{J} + \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$

These are known as **Maxwell’s equations.**

We will use them in their simplest possible form which is in free space where $\rho = 0$ and $\mathbf{J} = 0$. The equations are now expressed as:

$$\mathrm{div}\mathbf{E} = 0 \quad \_\_\_\_\_ \quad (1) \qquad\qquad \mathrm{curl}\mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \quad \_\_\_\_\_ \quad (3)$$

$$\mathrm{div}\mathbf{H} = 0 \quad \_\_\_\_\_ \quad (2) \qquad\qquad \mathrm{curl}\mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \_\_\_\_\_ \quad (4)$$

Now use the vector identity: $\quad \mathrm{curl}\,\mathrm{curl}\mathbf{E} = \mathrm{grad}\,\mathrm{div}\mathbf{E} - \nabla^2\mathbf{E}$

and rearrange it as: $\quad \nabla^2\mathbf{E} = \mathrm{grad}\,\mathrm{div}\mathbf{E} - \mathrm{curl}\,\mathrm{curl}\mathbf{E}$

Using equations (1) and (3): $\quad \nabla^2\mathbf{E} = \mathrm{curl}\left(\mu_0 \dfrac{\partial \mathbf{H}}{\partial t}\right) = \mu_0 \dfrac{\partial}{\partial t}(\mathrm{curl}\mathbf{H})$

and from (4): $\quad \nabla^2\mathbf{E} = \mu_0 \dfrac{\partial}{\partial t}\left(\varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}\right) = \mu_0\,\varepsilon_0 \dfrac{\partial^2 \mathbf{E}}{\partial t^2}$

Similarly, we can obtain: $\quad \nabla^2\mathbf{H} = \mu_0\,\varepsilon_0 \dfrac{\partial^2 \mathbf{H}}{\partial t^2}$

These are of the same mathematical form as the wave equation $\nabla^2\phi = \dfrac{1}{v^2}\dfrac{\partial^2\phi}{\partial t^2}$

Amazingly, we have discovered waves in electric and magnetic fields (actually, Maxwell got there first), which propagate at a speed of $1/\sqrt{\mu_0\,\varepsilon_0}$ m / s.

With $\mu_0 = 4\pi \times 10^{-7}$ henrys / metre and $\varepsilon_0 = 8.85419 \times 10^{-12}$ farads / metre, we find that the speed of the waves is 2.99792×10^8 m / s, a number which you ought to recognise as the speed of light. Mathematical physics doesn’t come any better than this!

I hope you have enjoyed the course.